INCOMPLETE GAUSS SUMS

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Abstract. Let N be a positive integer. We are concerned with the sum

$$G_N(m) = \sum_{j=0}^{m-1} e^{\{2\pi i j^2/N\}}.$$

Thus $G_N(N)$ is the ordinary Gauss sum. Previous methods of estimating such exponential sums have not brought to light the peculiar behaviour of $G_N(m)$ for $m < N/2$, namely that, for almost all values of m, $G_N(m)$ is in the vicinity of the point $\sqrt{N}(1+i)/4$. A sharp estimate is given for max $|G_N(m)|$, depending on the residue of N modulo 4. The results were suggested by graphs of $G_N(m)$ made for N near 1000. The analysis employs the Fresnel integrals and the Cornu spiral whose curvature is proportional to its arc length.

1. *Introduction.* If N is a positive integer, we call the exponential sum

$$G_N(m) = \sum_{v=0}^{m-1} e^{\{2\pi i v^2/N\}} \tag{1}$$

an incomplete Gauss sum in the case when $m < N$. The complete Gauss sum $G_N(N)$ is well known to be

$$G_N(N) = \begin{cases} (1+i)\sqrt{N}, & \text{if } N = 4k, \\ \sqrt{N}, & \text{if } N = 4k+1, \\ 0, & \text{if } N = 4k+2, \\ i\sqrt{N}, & \text{if } N = 4k+3. \end{cases} \tag{2}$$

If $m > N$ so that

$$m = qN + m_0 \ (0 \leqslant m_0 < N),$$

then clearly

$$G_N(m) = q G_N(N) + G_N(m_0).$$

Thus it is sufficient in studying the behaviour of $G_N(m)$ to consider the incomplete case. The best previous result appears to be that of Hua [1], which in this case becomes

$$|G_N(m)| = O(N^{\frac{1}{2}+\varepsilon}).$$

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Our aim in this paper is to give best possible estimates for $|G_N(m)|$ which hold uniformly in m when $m \leqslant N/2$ for the four residues of N (mod 4). In fact four constants c_r, $(r = 0, 1, 2, 3)$ are found such that

$$M_N = \max_{m \leqslant (N+1)/2} |G_N(m)| = c_r N^{\frac{1}{2}} + O(1) \qquad (N \equiv r \pmod 4).$$

That it suffices to treat the cases $m \leqslant (N+1)/2$ follows from the identity

$$G_N(m) + G_N(N - m + 1) = 1 + G_N(N). \tag{3}$$

These results are the consequences of realizing, from graphic experiments with large N, that almost all the values of $G_N(m)$, for N fixed and $m < N/2$, lie in the neighbourhood of a single point in the complex plane, namely $\sqrt{N}(1+i)/4$. To illustrate this phenomenon we define the *graph* of the sequence

$$G_N(0), \ G_N(1), \ \dots, \ G_N(N). \tag{4}$$

This directed graph has for vertices the successive terms of (4) and for edges the unit vectors joining $G_N(m)$ to $G_N(m+1)$ directed from the former to the latter. Thus the total length of the graph is N. We show in Figures 1, 2, 3, 4 these graphs for $N = 1024$, 1009, 1026 and 1155 respectively. In each figure, O marks the origin in the complex plane. In the case of Figure 3 the graph is doubly traversed.

Of course these graphs are in reality piecewise linear, not smooth curves with continuously turning tangents. However, we shall see that for $m = O(N^{\frac{1}{2}})$ the first m edges closely approximate a clothoid or Cornu spiral whose curvature is proportional to its arclength. This spiral has infinite length and so this comparison of the graph to the spiral is valid and useful only up to a certain point.

2. *Three Lemmas*. We use a rudimentary form of the Euler–Maclaurin summation formula which we develop in three lemmas.

LEMMA 1. *Let $y(x)$ be real, continuous, and monotone on the interval $0 \leqslant x \leqslant m$.* *Then*

$$\left| \int_0^1 (x^2 - x + \tfrac{1}{6}) \sum_{v=0}^{m-1} y(v+x) \, dx \right| \leqslant \frac{\sqrt{3}}{54} |y(m) - y(0)|.$$

Proof. Let $\rho_1 = (3 - \sqrt{3})/6$, $\rho_2 = (3 + \sqrt{3})/6$, so that

$$x^2 - x + 1/6 = (x - \rho_1)(x - \rho_2),$$

and let

$$w(t) = \int_0^t (x^2 - x + \tfrac{1}{6}) \, dx = t(t-1)(2t-1)/6.$$

Then

$$w(0) = w(1) = 0, \qquad w(\rho_1) = -w(\rho_2) = \sqrt{3}/108.$$

Without loss of generality we may take y to be monotone increasing. Let

$$T_v = \int_0^1 (x^2 - x + \tfrac{1}{6}) \, y(v+x) \, dx.$$

FIG. 1   FIG. 2
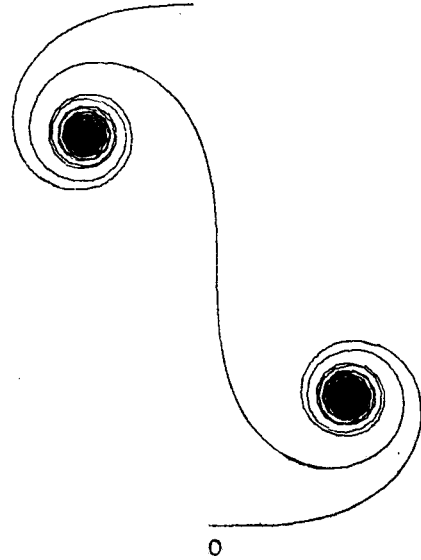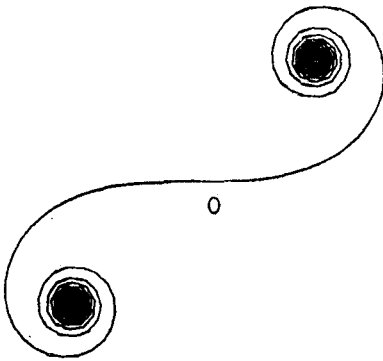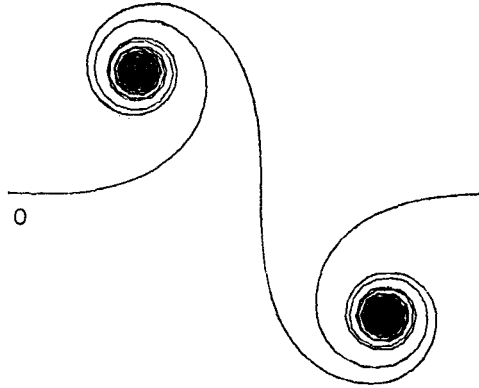

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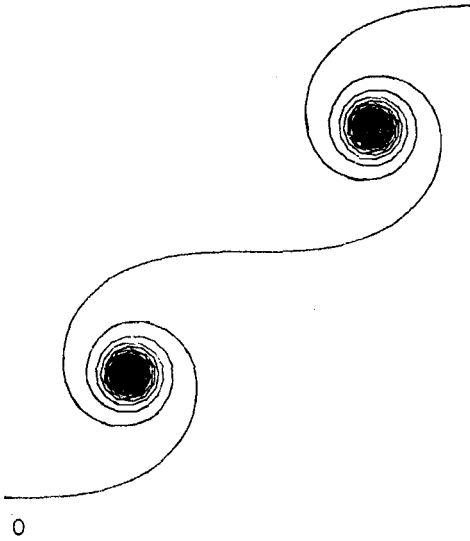
FIG. 3   FIG. 4

Then, taking into account the sign of $x^2 - x + \frac{1}{6}$ in $[0, 1]$,

$$T_v \leqslant y(v+1)\,\{w(\rho_1) - w(0)\} + y(v)\,\{w(\rho_2) - w(\rho_1)\} + y(v+1)\,\{w(1) - w(\rho_2)\}$$

$$= \{w(\rho_1) - w(\rho_2)\}\,\{y(v+1) - y(v)\}$$

$$= \frac{\sqrt{3}}{54}\,\{y(v+1) - y(v)\}.$$

Summing over v gives the lemma.

In the following two lemmas we require the identity

$$I = \int_0^1 (x^2 - x + \tfrac{1}{6}) \sum_{\nu=p}^{q-1} f''(\nu + x)\, dx$$

$$= \tfrac{1}{6}\{f'(q) - f'(p)\} - 2 \sum_{\mu=p}^{q} f(\mu) + f(p) + f(q) + 2 \int_p^q f(t)\, dt, \tag{5}$$

which is easily obtained by integrating twice by parts.

LEMMA 2. *Let f be a real function for which $f''(x)$ is monotone on the interval $p \leqslant x \leqslant q$. Then*

$$\sum_{\mu=p}^{q} f(\mu) = \int_p^q f(t)\, dt + \tfrac{1}{2} f(p) + \tfrac{1}{2} f(q) + \tfrac{1}{12}\{f'(q) - f'(p)\} + R$$

with

$$|R| \leqslant \frac{\sqrt{3}}{108} |f''(q) - f''(p)|.$$

Proof. Inspection of (5) reveals that

$$R = -\tfrac{1}{2} I. \tag{6}$$

Applying Lemma 1 with $y(x) = f''(p + x)$ and $m = q - p$ gives

$$|I| \leqslant \frac{\sqrt{3}}{54} |f''(q) - f''(p)|,$$

so the lemma follows from (6). It does not seem to be in the literature [2].

LEMMA 3. *Let f be such that $f''(x)$ is monotone decreasing for $p \leqslant x < \xi$ and monotone increasing for $\xi \leqslant x \leqslant q$. Let $n = [\xi]$ and let M be the larger of $(f''(n) - f''(\xi))$ and $f''(n+1) - f''(\xi)$. Then the R of Lemma 2 satisfies*

$$|R| \leqslant \frac{\sqrt{3}}{108} \{f''(p) + f''(q) - f''(n+1) - f''(n)\} + \tfrac{1}{24} M. \tag{7}$$

Proof. The I in (5) can be written $I = I_1 + I_2 + I_3$ where

$$I_1 = \int_0^1 (x^2 - x + \tfrac{1}{6}) \sum_{\nu=p}^{n-1} f''(\nu + x)\, dx,$$

$$I_2 = \int_0^1 (x^2 - x + \tfrac{1}{6}) f''(n + x)\, dx,$$

$$I_3 = \int_0^1 (x^2 - x + \tfrac{1}{6}) \sum_{\nu=n+1}^{q-1} f''(\nu + x)\, dx.$$

By Lemma 1,

$$|I_1| \leqslant \frac{\sqrt{3}}{54}(f''(p)-f''(n)), \qquad |I_3| \leqslant \frac{\sqrt{3}}{54}(f''(q)-f''(n+1)).$$

Since $f''(x)$ has a minimum at $x = \xi$,

$$0 \leqslant f''(n+x)-f''(\xi) < M \quad (0 \leqslant x \leqslant 1).$$

Hence, if we set

$$f''(n+x) = f''(\xi)+\tfrac{1}{2}M+h(x),$$

then

$$|h(x)| \leqslant M/2, \qquad 0 \leqslant x \leqslant 1. \quad \sim$$

Hence

$$I_2 = \int\limits_0^1 (x^2-x+\tfrac{1}{6})\,h(x)\,dx+\{f''(\xi)+\tfrac{1}{2}M\} \int\limits_0^1 (x^2-x+\tfrac{1}{6})\,dx,$$

and since

$$\int\limits_0^1 (x^2-x+\tfrac{1}{6})\,dx = 0 \quad \text{and} \quad |x^2-x+\tfrac{1}{6}| \leqslant \tfrac{1}{6}$$

we have

$$|I_2| \leqslant M/12.$$

The lemma now follows from the fact that

$$|R| = \tfrac{1}{2}|I| \leqslant \tfrac{1}{2}\{|I_1|+|I_2|+|I_3|\}.$$

3. *Notation and Normalization.* Instead of studying $G_N(m)$ as defined by (1), we find it simpler to consider a normalized version $g_N(m)$ defined by

$$g_N(m) = 2N^{-\frac{1}{2}} \sum_{v=0}^m e^{\{2\pi i v^2/N\}}. \tag{8}$$

The " scale factor " $2/\sqrt{N}$ does not change the aspects of the graphs depicted in Figures 1, 2, 3, 4. The length of each edge is now $2N^{-\frac{1}{2}}$ and for all N the complete graph $g_N(N-1)$ can now be covered by a disk of radius $\sqrt{8}$.

The relation (3) now becomes, in view of (2),

$$g_N(m)+g_N(N-m-1) = 2N^{-\frac{1}{2}}+ \begin{cases} 2+2i, & N = 4k, \\ 2, & N = 4k+1, \\ 0, & N = 4k+2, \\ 2i, & N = 4k+3. \end{cases}$$

From this we can find the exact values of $g_N(m)$ at the midpoint of its graph, as follows:

$$g_{4k}(2k-1) = 1+i, \tag{9}$$

$$g_{4k+1}(2k) = 1+N^{-\frac{1}{2}}, \tag{10}$$

$$g_{4k+2}(2k+1) = 0, \tag{11}$$

$$g_{4k+3}(2k+1) = i+N^{-\frac{1}{2}}. \tag{12}$$

We use the following notation and facts concerning the Fresnel integrals and the Cornu spiral. For the former we use

$$C(s) = \int_0^s \cos(\pi u^2/2)\, du, \qquad S(s) = \int_0^s \sin(\pi u^2/2)\, du. \qquad (13)$$

For the spiral we take the curve whose parametric equations are

$$x = C(s), \qquad y = S(s) \qquad (-\infty < s < \infty).$$

The choice of the letter s as a parameter is appropriate, since

$$(dx)^2 + (dy)^2 = \{(\cos(\pi s^2/2))^2 + (\sin(\pi s^2/2))^2\}\,(ds)^2 = (ds)^2,$$

so that s is the actual arclength distance of the point (x, y) from the origin of coordinates. If ψ is the angle through which the tangent line at the point $P : (x, y)$ has turned since P left the origin, we have

$$\tan \psi = dy/dx = \sin(\pi s^2/2)/\cos(\pi s^2/2) = \tan(\pi s^2/2),$$

so the intrinsic equation of our Cornu spiral is $\psi = \pi s^2/2$. Its radius of curvature ρ is given by

$$\rho = \frac{ds}{d\psi} = \frac{ds}{\pi s\, ds} = (\pi s)^{-1}.$$

Thus the curvature at P is π times the arclength distance of P from the origin.

4. *The case $m = O(N^{\frac{1}{4}})$.* We begin by examining $g_N(m)$ for $m = O(N^{\frac{1}{4}})$. To this effect we define

$$\omega = \omega(x) = 2\pi x^2/N, \qquad F_1(x) = \cos \omega, \qquad F_2(x) = \sin \omega,$$

so that

$$F_1'(x) = -4\pi x N^{-1} \sin \omega, \qquad (14)$$

$$F_2'(x) = 4\pi x N^{-1} \cos \omega, \qquad (15)$$

$$F_1''(x) = -4\pi N^{-1} \{\sin \omega + 2\omega \cos \omega\}, \qquad (16)$$

$$F_2''(x) = 4\pi N^{-1} \{\cos \omega - 2\omega \sin \omega\}, \qquad (17)$$

$$F_1'''(x) = -(4\pi N^{-1})^2 \, x\{3 \cos \omega - 2\omega \sin \omega\},$$

$$F_2'''(x) = -(4\pi N^{-1})^2 \, x\{3 \sin \omega + 2\omega \cos \omega\},$$

$$F_1^{(4)}(x) = (4\pi N^{-1})^2 \{12\omega \sin \omega + (4\omega^2 - 3) \cos \omega\},$$

$$F_2^{(4)}(x) = (4\pi N^{-1})^2 \{- 12\omega \cos \omega + (4\omega^2 - 3)\sin \omega\}.$$

Let $\omega_1 = 0{\cdot}98824073$ and $\omega_2 = 2{\cdot}17462603$ be the least positive solutions of

$$\cot \omega = 2\omega/3 \quad \text{and} \quad \tan \omega = -2\omega/3,$$

and set

$$L_1 = (N\omega_1/2\pi)^{\frac{1}{2}} = 0{\cdot}39658971 \,\sqrt{N},$$

$$L_2 = (N\omega_2/2\pi)^{\frac{1}{2}} = 0{\cdot}58830475 \,\sqrt{N},$$

$$L_3 = (N/2)^{\frac{1}{2}} \quad\ = 0{\cdot}70710678 \,\sqrt{N}.$$

Then

$$F_1''(x) \text{ is monotone decreasing for } 0 \leqslant x \leqslant L_1,$$

$$F_1''(x) \text{ is monotone increasing for } L_1 \leqslant x \leqslant L_3,$$

$$F_2''(x) \text{ is monotone decreasing for } 0 \leqslant x \leqslant L_2,$$

$$F_2''(x) \text{ is monotone increasing for } L_2 \leqslant x < L_3.$$

We now have

THEOREM 1. *If $m \leqslant L_3$ and $N \geqslant 100$, then*

$$\left| \sum_{v=0}^{m} \cos(2\pi v^2/N) - \int_0^m \cos(2\pi t^2/N)\,dt \right| < \underset{\sim}{1} + N^{-\frac{1}{4}},$$

$$\left| \sum_{v=0}^{m} \sin(2\pi v^2/N) - \int_0^m \sin(2\pi t^2/N)\,dt \right| < \tfrac{1}{2} + N^{-\frac{1}{4}}.$$

Proof. First suppose $m \leqslant L_1$. We apply Lemma 2 to the case of $f(x) = F_1(x)$, $p = 0, q = m$. We then conclude that in view of (14) and (16)

$$\left| \sum_{v=0}^{m} \cos(2\pi v^2/N) - \int_0^m \cos(2\pi t^2/N)\,dt \right|$$

$$< \tfrac{1}{2} + \tfrac{1}{2} + \frac{4\pi m}{12N} + \frac{\sqrt{3}}{108}\left\{ \frac{4\pi}{N}\left(1 + \frac{4m^2\pi}{N}\right)\right\}$$

$$< 1 + \tfrac{5}{12}N^{-\frac{1}{4}} + \tfrac{2}{3}N^{-1} < 1 + N^{-\frac{1}{4}},$$

since $m \leqslant L_1$ and $N > 100$.

Next suppose $L_1 \leqslant m \leqslant L_3$. We apply Lemma 3 to the case of $f(x) = F_1(x)$, $p = 0, \xi = L_1, q = L_3$. Since ξ is in this case a function of N it is difficult to specify $n = [\xi]$. So we write

$$f''(p) + f''(q) - f''(n) - f''(n + 1) < f''(p) + f''(q) - 2f''(\xi),$$

and use the facts that

$$f''(n) - f''(\xi) < f''(\xi - 1) - f''(\xi)$$

$$f''(n + 1) - f''(\xi) < f''(\xi + 1) - f''(\xi)$$

and that $f''(\xi \pm 1) - f''(\xi) = (1/2!)f^{(4)}(\xi) \pm f^{(5)}(\xi + \theta)/6$. That is we may take $M < (1/2!)f^{(4)}(\xi) < 814/N^2$. The application of Lemma 3 now gives

$$\left| \sum_{v=0}^{m} \cos(2\pi v^2/N) - \int_0^m \cos(2\pi t^2/N)\,dt \right|$$

$$< \tfrac{1}{2} + \tfrac{1}{2} + \frac{4\pi m}{12N} + \frac{\sqrt{3}}{108}\{F_1''(m) - 2F_1''(L_1)\} + \frac{34}{N^2}$$

$$< 1 + \frac{\pi\sqrt{2}}{6}N^{-\frac{1}{4}} + \frac{118}{N}\frac{34}{N^2} < 1 + N^{-\frac{1}{4}},$$

since $m \leqslant L_3$ and $N > 54$.

This proves the first inequality of the theorem. For the second inequality, if $m < L_2$, Lemma 2 gives us with $f(x) = F_2(x)$, $p = 0$, $q = m$, using (15) and (17),

$$\left| \sum_{v=0}^{m} \sin (2\pi v^2/N) - \int_0^m \sin (2\pi t^2/N)\, dt \right|$$

$$< \tfrac{1}{2} + \frac{\pi m}{3N} + \frac{\sqrt{3}}{108} \frac{4\pi}{N} (2 + 4\pi m^2/N)$$

$$< \tfrac{1}{2} + 0 \cdot 6161 N^{-\frac{1}{2}} + 0 \cdot 8766 N^{-1}$$

$$< \tfrac{1}{2} + N^{-\frac{1}{2}}$$

since $N > 100$.

Next suppose $L_2 < m < L_3$. Applying Lemma 3 as before, we get

$$\left| \sum_{v=0}^{m} \sin (2\pi v^2/N) - \int_0^m \sin (2\pi t^2/N)\, dt \right|$$

$$< \tfrac{1}{2} + 0 \cdot 7405 N^{-\frac{1}{2}} + 1 \cdot 672 N^{-1} + 91 N^{-2}$$

$$< \tfrac{1}{2} + N^{-\frac{1}{2}},$$

since $N > 100$. This completes the proof of Theorem 1.

Combining Theorem 1 with the normalization and notation of (8) and (13), we have

THEOREM 2. *If* $m \leqslant \sqrt{(N/2)}$ *and* $N \geqslant 100$, *then*

$$|g_N(m) - C(2mN^{-\frac{1}{2}}) - iS(2mN^{-\frac{1}{2}})| < (101/40) N^{-\frac{1}{2}}.$$

Proof. By Theorem 1, there exist θ_1 and θ_2, both less than 1 in absolute value, such that

$$\sum_{v=0}^{m} e^{2\pi i v^2/N} - \int_0^m e^{2\pi i t^2/N}\, dt = \theta_1 (1 + N^{-\frac{1}{2}}) + i\theta_2 (\tfrac{1}{2} + N^{-\frac{1}{2}}) = \Delta. \qquad (18)$$

We have $|\Delta|^2 \leqslant \tfrac{5}{4}(1 + \tfrac{12}{5} N^{-\frac{1}{2}} + \tfrac{8}{5} N^{-1})$, so that

$$|\Delta| < \sqrt{(5)}/2 \,(1 + \tfrac{6}{5} N^{-\frac{1}{2}} + \tfrac{4}{5} N^{-1}) < 101/80,$$

since $N \geqslant 100$. If we multiply (18) by $2N^{-\frac{1}{2}}$ and take absolute values of both sides, replacing t by $\tfrac{1}{2}N^{\frac{1}{2}}u$, we get the theorem.

5. *The case* $\sqrt{(N/2)} < m < N/4$. Having considered the early part of the graph of $g_N(m)$, that is for $m \leqslant \sqrt{(N/2)}$, we can now locate the rest of the graph. We begin with

THEOREM 3. *Let* Γ *be the circle with centre at the point*

$$Q : (C(\sqrt{2}), S(\sqrt{2}) - (\sqrt{2}\pi)^{-1}) = (0 \cdot 5288892, 0 \cdot 4888933)$$

and radius $1/\sqrt{2(\pi)} + 101/40\, N^{-\frac{1}{2}}$. *Then for* $N \geqslant 100$, *that part of the graph of* $g_N(m)$ *for which* $\sqrt{(N/2)} \leqslant m \leqslant N/4$ *lies within* Γ.

Proof. Let Γ' be the circle of curvature of the Cornu spiral at the point P where $s = \sqrt{2}$. Here $\psi = \pi$ so the tangent at P is horizontal. The radius of Γ' is

$$\rho = (\sqrt{2}\pi)^{-1}$$

and so the centre of Γ' is Q.

In going from the vertex $g_N(m)$ to the next vertex

$$g_N(m+1) = g_N(m) + e^{2\pi i(m+1)^2/N}$$

the new edge suffers a rotation, with respect to the previous edge, of

$$\delta(m) = 2\pi/N \{(m+1)^2 - m^2\}^n = 2\pi/N (2m+1).$$

For brevity we call this angle the *departure* of the mth vertex of the graph.

Let $m_0 = [\sqrt{(N/2)}]$. Then

$$\delta(m_0) \sim \sqrt{(8)}\, \pi N^{-\frac{1}{2}},$$

and by Theorem 2 the vertex $g_N(m_0)$ is within the distance $(101/40)N^{-\frac{1}{2}}$ of the point P on the spiral, so this vertex is inside Γ. If further departures $\delta(m)$ for $m > m_0$ did not increase steadily to π, but instead remained fixed at $\delta(m_0)$, the graph would just barely remain inside Γ, staying near its circumference. As it is, the successive edges of the graph turn more and more inward until the graph is oscillating across a small circle whose diameter is about $2N^{-\frac{1}{2}}$, the common edge length of the graph. Hence for $\sqrt{(N/2)} \leqslant m \leqslant N/4, g_N(m)$ lies within Γ.

The behaviour of the graph $g_N(m)$ for $N/4 \leqslant m \leqslant N/2$ is, to put it roughly, an unwinding of the first quarter of the graph. To discuss this it is best to consider separately the four cases of N modulo 4. We assume that from now on $N \geqslant 100$.

6. The cases $N = 4k + j, j = 0, 2, 3$.

THEOREM 4. *If* $N = 4k + 2$ *then for* $m \leqslant N$

$$|g_N(m)| < 0.9490569 + (101/40)\, N^{-\frac{1}{2}}.$$

Proof. In this case we have for the departure of the mth vertex

$$\delta(m) = \begin{cases} \pi + \dfrac{2\pi r}{2k+1}, & \text{if } m = k + r, \\[3mm] \pi - \dfrac{2\pi r}{2k+1}, & \text{if } m = k - r. \end{cases}$$

This means that after m reaches $k = (N - 2)/4$ the graph $g_N(m)$ exactly retraces itself until it reaches the origin. The maximum distance from the origin of a point on the Cornu spiral is 0.9490569 which is achieved at $s = 1.2093781$ where $\psi = 2.297439$ and $\rho = 0.2632013$. Applying the inequality of Theorem 2 completes the proof.

THEOREM 5. *If* $N = 4k + 3$ *and* $m < N/2$, *then*

$$\max |g_N(m)| = \sqrt{(1 + N^{-1})},$$

and this is achieved for $m = 2k + 1$.

Proof. By (12),

$$|g_{4k+3}(2k+1)| \leqslant \sqrt{(1+N^{-1})}.$$

This is greater than maximum distance 0·9490569 of the incoming spiral, $m \leqslant N/4$.

THEOREM 6. *If $N = 4k$ and $m \leqslant N/2$, then*

$$\max |g_N(m)| = \sqrt{2}.$$

Proof. This follows from (9).

7. *The case $N = 4k + 1$.* This case is a much more difficult one. One can see from Figure 2 that there is a point $g_N((N-1)/2 - m)$ for $m = O(N^{-\frac{1}{4}})$ at a maximum distance from the origin. By (8) we have

$$g_{4k+1}(2k-m) = g_{4k+1}(2k) - N^{-\frac{1}{2}} \sum_{v=0}^{m} e^{2\pi i(2k-j)^2/N}$$

$$= 1 + \frac{1}{\sqrt{N}} - \frac{2}{\sqrt{N}} e^{-2\pi ik/N} \sum_{v=0}^{m} e^{2\pi i(j^2+j)/N}.$$

Applying Lemmas 2 and 3 to the functions

$$\cos\{2\pi(x^2+x)/N\} \quad \text{and} \quad \sin\{2\pi(x^2+x)/N\},$$

we show that this last sum is approximated by the integral

$$\int_0^m e^{2\pi i(t^2+t)/N} \, dt.$$

Setting $t + \frac{1}{2} = \frac{1}{2}\sqrt{(N)}u$ and noting that

$$e^{-2\pi ik/N} = -i(1 + O(N^{-1})),$$
$$C(N^{-\frac{1}{4}}) = O(N^{-\frac{1}{4}}),$$
$$S(N^{-\frac{1}{4}}) = O(N^{-\frac{3}{4}}),$$

we find that for $m = O(N^{-\frac{1}{4}})$

$$g_{4k+1}(2k-m) = 1 - S(2m+1/N^{\frac{1}{2}}) - iC(2m+1/N^{\frac{1}{2}}) + O(N^{-\frac{1}{4}}).$$

This shows that this part of the graph closely follows a Cornu spiral centred at $(1+i)/2$ but rotated through $\pi/2$, and so $g_{4k+1}(2k-m)$ for $\sqrt{(N)}/2 < m < N/4$ lies inside a circle of radius $(\sqrt{(2)}\pi)^{-1} + O(N^{-\frac{1}{4}})$ as before. To find the point at the greatest distance from the origin we must make

$$\{1 - S(u)\}^2 + \{C(u)\}^2$$

a maximum. This is achieved for $u = 0\cdot6136886$ and is

$$1\cdot12900425 = (1\cdot0625461)^2.$$

Hence we have

THEOREM 7. *If $N = 4k + 1$ and $m < N/2$, then*

$$\max_{0 < m < 2k} |g_N(m)| = 1 \cdot 0625461 + O(N^{-\frac{1}{4}}),$$

and this is achieved for

$$m = [2k + \tfrac{1}{2} - 0 \cdot 3068442\sqrt{N}].$$

We have spared the printer and the reader the details of the proof of the above theorem. If the reader wishes to apply the above methods he will find the tables of the Fresnel integrals by the Russian Academy [3] very useful.

References

1. Hua, Loo Keng. " On exponential sums ", *Sci. Record* (new series), 1 (1957), 1–4 (*Math. Revs.* 20, 22).
2. An authoritative account of the Euler–Maclaurin summation formula will be found in A. M. Ostrowski, " On the remainder term of the Euler–Maclaurin formula ", *Jn. Math. reine angew*, 239–240 (1969), 268–286.
3. *Tables of Fresnel Integrals* (Akad. Nauk. SSSR. Moscow 1953).

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